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2001 J. Phys. A: Math. Gen. 34 2415

(http://iopscience.iop.org/0305-4470/34/11/331)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2415–2433 www.iop.org/Journals/ja PII: S0305-4470(01)15477-6

Integrability in the theory of geodesically equivalent metrics

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Received 2 June 2000, in final form 11 January 2001

Abstract
The property of two metrics on one manifold having the same geodesics is equivalent to a special kind of integrability of the geodesic flows of these metrics (both in the classical and in the quantum sense). This gives us nontrivial restrictions on the topology of the manifold, allows us to construct new examples of such pairs of metrics on the sphere and to give a local description of geodesically equivalent metrics near the points where the eigenvalues of one metric with respect to the other bifurcate.

PACS numbers: 0240G, 0240H, 0240K, 0240U, 0240X, 0240Y, 0230I, 0230O, 0230T, 0365G, 0365D

1. Introduction

Definition 1. *Riemannian metrics g and $\bar{g}$ on M^n are geodesically equivalent, if their geodesics coincide (as unparametrized curves).*

Metrics with the same geodesics were considered by Beltrami [1]. The list of mathematicians who worked on this subject includes Dini, Levi-Civita, Liouville, Painleve, Weyl, Cartan and Eisenhart. Once mathematicians understood what geodesics of Riemannian metrics were, it was very natural to look for two different metrics having the same geodesics.

Since the time of Beltrami, the main technique for investigation of geodesically equivalent metrics has been based on the following PDE system: in tensor notations, the equation for $\bar{g}$ to be geodesically equivalent to g is as follows:

$$2(n+1)\bar{g}_{ij,k} = 2\bar{g}_{ij}\Theta_{,k} + \bar{g}_{ik}\Theta_{,j} + \bar{g}_{kj}\Theta_{,i}. \qquad (1)$$

Here Θ denotes the function $\ln(\frac{\det(\bar{g})}{\det(g)})$ and $T_{,l}$ is the covariant derivative of the tensor T with respect to the metric g.

The system (1) is nonlinear; it is almost impossible to solve it globally or for a given non-trivial metric g; the most remarkable local result is probably the local description of geodesically equivalent metrics given by Dini in [6] for surfaces and by Levi-Civita in [10] for manifolds of arbitrary dimension. Here we formulate the Levi-Civita theorem assuming that all eigenvalues of one metric with respect to the other are different. Denote by G the tensor $g^{i\alpha}\bar{g}_{\alpha j}$.

Theorem 1 (Levi-Civita [10]). *Let g, $\bar{g}$ be geodesically equivalent on M^n. Suppose that at the point x all eigenvalues of G are different and equal to $\rho^1(x) > \cdots > \rho^n(x)$. Then there exists a coordinate system $x_1, \ldots, x_n$ in some neighbourhood U^n of the point x such that the quadratic forms of the metrics g and $\bar{g}$ have the following form:*

$$\mathrm{d}g^2 = \Pi_1\,\mathrm{d}x_1^2 + \cdots + \Pi_n\,\mathrm{d}x_n^2 \tag{2}$$

$$\mathrm{d}\bar{g}^2 = \rho^1\Pi_1\,\mathrm{d}x_1^2 + \cdots + \rho^n\Pi_n\,\mathrm{d}x_n^2 \tag{3}$$

where the functions Π_i and ρ_i are given by

$$\Pi_i \stackrel{\text{def}}{=} (\phi_i - \phi_1)\cdots(\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i)\cdots(\phi_n - \phi_i)$$

$$\rho^i \stackrel{\text{def}}{=} \frac{1}{\phi_1\ldots\phi_n}\frac{1}{\phi_i}$$

where $\phi_1 < \cdots < \phi_n$ are smooth functions on U^n such that for any i the function ϕ_i depends on the variable x_i only.

Remark 1. The metrics (2), (3) are geodesically equivalent.

We generalize this theorem in section 6 to a case of the points on M^n where the eigenvalues of the operator G bifurcate.

2. Integrability

The goal of this paper is to give a review of new results on geodesically equivalent metrics. The results are global, in the sense that the manifold is assumed to be closed (or at least geodesically connected). All previously known global results on geodesically equivalent metrics require additional strong geometrical assumptions. For example, for Einstein or (hyper)Kahlerian metrics beautiful results were obtained by Lichnerowicz [14], Venzi [13], Mikes [23], Couty [5] and Hasegawa and Fujimura [8].

Our methods are also useful for local description of geodesically equivalent metrics. We use it to obtain generalization of Levi-Civita's theorem to the case when the eigenvalues of the tensor G bifurcate.

The new approach to geodesically equivalent metrics were suggested in [29]. Essentially, it was shown that the theory of geodesically equivalent metrics can be considered as a part of the theory of finite-dimensional integrable systems: using $\bar{g}$ we can construct invariantly n commuting integrals for the geodesic flow of g.

Let g, $\bar{g}$ be Riemannian metrics on M^n. Consider the tensor $G^i_j = g^{i\alpha}\bar{g}_{\alpha j}$ which in invariant terms can be given as the fibrewise-linear mapping $G : TM^n \to TM^n$ such that $g(G(\xi), \nu) = \bar{g}(\xi, \nu)$ for any $x \in M^n$ and for any $\xi, \nu \in T_xM^n$.

Consider the characteristic polynomial $\det(G - \mu\mathbf{1}) = c_0\mu^n + c_1\mu^{n-1} + \cdots + c_n$, where $\mathbf{1}$ denotes the identity. Consider the mappings $S_0, S_1, \ldots, S_{n-1} : TM^n \to TM^n$ given by

$$S_k \stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}.$$

Consider the functions $I_0, I_1, \ldots, I_{n-1} : T^*M^n \to R$ given by the general formula

$$I_k(x, p) \stackrel{\text{def}}{=} g^{\alpha j}(S_k)^i_\alpha p_i p_j. \tag{4}$$

In invariant terms, if we identify TM^n with T^*M^n by the metric g, the functions I_k are given by $I_k(x, \xi) = g(S_k\xi, \xi)$.

Remark 2. The integral I_{n-1} is the energy integral (multiplied by minus two).

Theorem 2 ([29]). *If g and $\bar{g}$ are geodesically equivalent then the functions I_k are commutative integrals for the geodesic flow of the metric g.*

As it has been shown in [17], if the metrics are not proportional at least at one point then at least one integral (namely, the integral I_0), is functionally independent of the Hamiltonian. In particular, if the geodesic flow of some metric g is ergodic then the following two statements are equivalent:

- $\bar{g}$ is geodesically equivalent to g.
- $\bar{g}$ is equal to Cg for an appropriate positive constant C.

Definition 2. *The metrics g, $\bar{g}$ are strictly non-proportional at $x_0 \in M^n$, if the characteristic polynomial* $\det(G - \mu\mathbf{1})$ *has no multiple roots at x_0.*

If the metrics are strictly non-proportional at $x \in M^n$ then the integrals I_k are functionally independent almost everywhere on the cotangent bundle to some neighbourhood of the point x. Corollary 1 shows that if the metrics are strictly non-proportional at one point of the manifold then they are strictly non-proportional almost everywhere and therefore the integrals are functionally independent almost everywhere.

Corollary 1 ([16,21]). *Suppose M^n is connected. Let metrics g, $\bar{g}$ on M^n be geodesically equivalent.*

If they are strictly non-proportional at least at one point of M^n then they are strictly non-proportional almost everywhere. More generally, if at a point of the manifold the number of different eigenvalues of G is equal to n_1 then at almost every point the number of different eigenvalues of G is greater than or equal to n_1.

3. Symplectic nature of the integrals

How is it possible to 'see' the integrals from theorem 2? Here we recall a construction (which is essentially due to [28]) that, given an orbital diffeomorphism between two Hamiltonian systems, produces integrals of them. Let v and $\bar{v}$ be Hamiltonian systems on symplectic manifolds (N, ω) and $(\bar{N}, \bar{\omega})$ with Hamiltonians H and $\bar{H}$, respectively. Consider the isoenergy surfaces

$$Q \stackrel{\text{def}}{=} \{x \in N : H(x) = h\} \qquad \bar{Q} \stackrel{\text{def}}{=} \{x \in \bar{N} : \bar{H}(x) = \bar{h}\}$$

where h and $\bar{h}$ are regular values of the functions H, $\bar{H}$, respectively.

Definition 3. *A diffeomorphism $\phi : Q \longrightarrow \bar{Q}$ is said to be orbital, if it takes the orbits (considered as unparametrized curves) of the system v to the orbits of the system $\bar{v}$.*

Given orbital diffeomorphism, we can invariantly construct integrals. Denote by σ, $\bar{\sigma}$ the restrictions of the forms ω, $\bar{\omega}$ to Q, $\bar{Q}$ respectively. Consider the form $\phi^*\bar{\sigma}$ on Q.

Lemma 1 ([28]). *The flow v preserves the form $\phi^*\bar{\sigma}$.*

It is obvious that the kernels of the forms σ and $\phi^*\bar{\sigma}$ coincide (in the space $\mathcal{T}_x Q$ at each point $x \in Q$) with the linear span of the vector v. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $\mathcal{T}Q/\langle v\rangle$. We shall denote the corresponding forms on $\mathcal{T}Q/\langle v\rangle$ also by the letters σ, $\bar{\sigma}$.

Lemma 2. *The characteristic polynomial of $(\sigma)^{-1}(\phi^*\bar{\sigma})$ on $\mathcal{T}Q/\langle v\rangle$ is preserved by the flow v.*

Since both forms are skew symmetric, each root of the characteristic polynomial $(\sigma)^{-1}(\phi^*\bar\sigma)$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n-1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow v. Therefore the coefficients of the polynomial $\delta^{n-1}(t)$ are integrals of the system v.

Geodesic flows of geodesically equivalent metrics can be considered as orbitally equivalent systems. The manifold $N = \bar N = TM^n$, the forms ω, $\bar\omega$ are given by

$$\omega \stackrel{\mathrm{def}}{=} \mathrm{d}[g_{ij}\xi^j\,\mathrm{d}x^i] \qquad \bar\omega \stackrel{\mathrm{def}}{=} \mathrm{d}[\bar g_{ij}\xi^j\,\mathrm{d}x^i]$$

and the orbital diffeomorphism ϕ is given by

$$\phi(x,\xi) = \left(x, \frac{\|\xi\|_g}{\|\xi\|_{\bar g}}\xi\right).$$

Direct calculations give us the formulae for the integrals I_k from theorem 2.

4. Integrability criterion

Does the Liouville integrability of the geodesic flow imply the existence of a geodesically equivalent metric? Let g, $\bar g$ be two metrics on M^n. Consider the functions $I_k : T^*M^n \to \boldsymbol{R}$, $k = 0, \ldots, n-1$, given by (4). Consider the standard symplectic structure on T^*M^n.

Theorem 3 ([21]). *Let the functions I_k commute and let them be functionally independent almost everywhere. Then the metrics g, $\bar g$ are geodesically equivalent.*

A more precise variant of this theorem is proved in [30].

Corollary 2. *Metrics g and $\bar g$ on a surface are geodesically equivalent, if and only if the function $(\frac{\det(g)}{\det(\bar g)})^{\frac{2}{3}}\bar g(\xi,\xi)$ is an integral of the geodesic flow of g.*

Metrics on surfaces with quadratically integrable geodesic flows were described in [9, 11], see also [2]. In view of corollary 2, this description gives us a complete description of geodesically equivalent metrics on surfaces.

5. The eigenvalues of the Sinjukov mapping are globally ordered

Here we give a proof of corollary 1, assuming that M^n is geodesically connected. Consider the fibrewise-linear mapping

$$A : TM^n \to TM^n \qquad A \stackrel{\mathrm{def}}{=} \det(G)^{\frac{1}{n+1}} G^{-1}. \tag{5}$$

The mapping A is called Sinjukov mapping, its characteristic invariants play an important role in the local theory of geodesically equivalent metrics. Denote by $\phi_1(x) \leqslant \cdots \leqslant \phi_n(x)$ the eigenvalues of A at $x \in M^n$.

Theorem 4 ([20]). *Let g, $\bar g$ be geodesically equivalent metrics on M^n. Suppose that M^n is geodesically complete (with respect to one of the metrics) and connected. Then for any $i \in \{1, \ldots, n-1\}$ the following statements are true:*

(1) $\phi_i(x) \leqslant \phi_{i+1}(y)$ for any $x, y \in M^n$.
(2) If $\phi_i(x) = \phi_{i+1}(x)$ for any point x of a neighbourhood $U \subset M^n$ then $\phi_i(y) = \phi_{i+1}(y) = \mathrm{const}$ for any point $y \in M^n$ (assuming that the constant const *is independent of the point).*
(3) If $\phi_i(x) = \phi_{i+1}(y)$ for some $x, y \in M^n$ then there exists $z \in M^n$ such that $\phi_i(z) = \phi_{i+1}(z)$.

The main idea of the proof is that the integrals I_k are given invariantly in local terms: the restriction of the integral I_k to each cotangent space $T_x^*M^n$ depends on the restriction of the metrics to the point x only. In particular, the next lemma is an exercise in finite-dimensional linear algebra and we leave the proof to the reader.

For each point $(x, \xi) \in T^*M^n$ (assuming that $x \in M^n$ and $\xi \in T_x^*M^n$), consider the following polynomial in t:

$$F_t(x, \xi) \stackrel{\text{def}}{=} t^{n-1} I_{n-1}(x, \xi) + t^{n-2} I_{n-2}(x, \xi) + \cdots + I_0(x, \xi). \tag{6}$$

The coefficients of this polynomial are functions on T^*M^n. Let us denote by $t_1(x, \xi) \leqslant \cdots \leqslant t_{n-1}(x, \xi)$ the roots of F_t.

Lemma 3 ([20]). *For any point $(x, \xi) \in T^*M^n$, for any $i \in \{1, \ldots, n-1\}$, the following statements are true:*

(1) $t_i(x, \xi)$ is real.
(2) $\phi_i(x) \leqslant t_i(x, \xi) \leqslant \phi_{i+1}(x)$.
(3) If for some open non-empty subset $V \subset T_x^*M^n$ the function t_i is constant on this subset then $\phi_i(x) = \phi_{i+1}(x)$.

Now we are able to prove the first two statements of theorem 4. By theorem 2, the coefficients I_k of the polynomial F_t are integrals of the geodesic flow of g. Then the roots t_i are constant on each orbit of the geodesic flow of g. By lemma 3, for any geodesic γ we have (we identify the tangent and the cotangent bundles of M^n by g)

$$\phi_i(\gamma(0)) \leqslant t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)) \leqslant \phi_{i+1}(\gamma(1)).$$

Since M^n is connected and geodesically complete, we can join each two points by a geodesic and the first statement of the theorem is proved.

Now let, for any point x of some neighbourhood $U \subset M^n$, the eigenvalue $\phi_i(x)$ be equal to $\phi_{i+1}(x)$. By the first statement of theorem 4 we then have that $\phi_i(x) = \phi_{i+1}(x) = \text{const}$, where the constant const is independent of $x \in U$. Take an arbitrary point $y \in M^n$. Let us joint the point y by all possible geodesics with every point of U. By the second statement of lemma 3, the value of t_i at each point of the corresponding geodesic orbits is equal to the constant const. Then the initial momenta of these geodesics form an open non-empty subset $V \subset T_y^*M^n$ and for any $\nu \in V$ the value $t_i(y, \nu)$ is equal to const. Thus by lemma 3 we have

$$\phi_i(y) = \phi_{i+1}(y) = t_i(y, \nu) = \text{const}$$

and the second statement of theorem 4 is proved.

Note that the second statement of theorem 4 already implies corollary 1, since the eigenvalues of G are evidently given by

$$\frac{1}{\phi_1(x)\ldots\phi_n(x)}\frac{1}{\phi_i(x)}.$$

Now let us explain the third statement of theorem 4. Suppose $\phi_i(x) = \phi_{i+1}(y) = \phi$. Consider a geodesic $\gamma : \boldsymbol{R} \to M^n$ such that $\gamma(0) = x$, $\gamma(1) = y$. By lemma 3, the value of t_i on the corresponding geodesic orbit $(\gamma, \dot\gamma)$ is equal to ϕ. The proof of the third part of the theorem consists of the following two statement, the complete proof of which is fairly lengthy and will appear elsewhere.

Statement 1. The differential of the function $F_\phi : T^*M \to \boldsymbol{R}$ (i.e. F_t for $t = \phi$) is zero at each point of the geodesic orbit $(\gamma, \dot\gamma)$.

Statement 2. If differential of the function F_ϕ is zero for some point $(z, \nu) \in T^*M^n$, $\nu \neq 0$, then either $\phi_i(z) = \phi$ or $\phi_{i+1}(z) = \phi$ (or $\phi_i(z) = \phi_{i+1}(z) = \phi$).

Since the geodesic γ is connected, and since the sets $\{w \in \gamma : \phi_i(w) = \phi\}$, $\{w \in \gamma : \phi_{i+1}(w) = \phi\}$ are closed and non-empty then they intersect so that there exists a point $z \in \gamma$ such that $\phi_i(z) = \phi_{i+1}(z) = \phi$, q.e.d.

6. Topology of the manifold with geodesically equivalent metrics

By theorem 2 and corollary 1, if two geodesically equivalent metrics on a connected manifold are strictly non-proportional at a point then their geodesic flows are completely integrable. This gives us a topological condition that prevents a closed manifold from possessing a pair of geodesically equivalent metrics that are strictly non-proportional at least at one point. The first versions of such a condition appeared in [17, 19, 29], here we present the last version.

Corollary 3. *Suppose M^n is connected and closed. Let metrics g, $\bar g$ on M^n be geodesically equivalent and strictly non-proportional at least at one point.*

Then M^n can be covered (with branched points) by the torus T^n.

The proof of corollary 3 is rather lengthy and will appear elsewhere; here we note that (in the typical case) the branched points of the covering are precisely the points where the metrics are not strictly non-proportional. If we lift the metrics to the covering torus then the resulting pseudo-Riemannian metrics have the Levi-Civita form (2), (3) in some global coordinate system on the torus. Below we show that this is also true locally: in the typical case, near the points where the metrics are not strictly non-proportional, there exists a Levi-Civita coordinate net with singularities of index $\frac{1}{2}$ in the points where the metrics are not strictly non-proportional.

In section 8 we will show that if at each point the metrics are strictly non-proportional then the manifold is covered by the torus without branch points, see theorem 9.

Conjecture 1. *Suppose M^n is connected and closed. Let metrics g, $\bar g$ on M^n be geodesically equivalent and strictly non-proportional at least at one point.*

Then the manifold M^n can be covered by the direct product of spheres.

It is possible to show that if each of the two manifolds admits a pair of geodesically equivalent metrics which are strictly non-proportional at least at a point then the direct products of the manifolds also admits a pair of geodesically equivalent metrics which are strictly non-proportional at least at a point. In section 8 we will show that any sphere admits a pair of geodesically equivalent metrics which are strictly non-proportional at least at a point, and therefore the product of spheres also admits a pair of geodesically equivalent metrics which are strictly non-proportional at least at a point.

Conjecture 2. *Suppose M^n is closed connected and hyperbolic (in the sense that it admits a metric of constant negative sectional curvature). Then if two metrics are geodesically equivalent on M^n then they are completely proportional: one metric equals the other multiplied by some constant.*

By corollary 3, we have that conjecture 2 is true for surfaces.

In what follows we give a detailed description of the (non-degenerate) branched points of the covering that appears in corollary 3. For this we need a more detailed analysis of the the set of the singular points of the pair of geodesically equivalent metrics.

Denote by $\mathcal{M}^q(g, \bar g)$ $(1 \leqslant q \leqslant n)$ the set of *the sable points of type q*, i.e., the set of the points $y \in M^n$ such that the operator $G|_x$ $(G \stackrel{\text{def}}{=} g^{-1}\bar g)$ has exactly q distinct eigenvalues for every x from an open neighbourhood of the point y. Denote by $\mathcal{M}(g, \bar{g})$ the set of all stable points on M^n, i.e., $\mathcal{M} \stackrel{\text{def}}{=} \bigsqcup_{i=1}^{n} \mathcal{M}^i$.

Definition 4. *A point $x \in M^n$ is called singular (with respect to the metrics g and $\bar{g}$) if x is not stable.*

Denote the set of the singular points by $\mathcal{S}$, $\mathcal{S} \stackrel{\text{def}}{=} M^n \setminus \mathcal{M}$.

Our aim is to find a local description of geodesically equivalent metrics in neighbourhoods of singular points.

Denote by $\mathcal{I}$ the vector space spanned on the integrals $I_0, \ldots, I_{n-1}$. It is proved in [31] that

$$I_c(g, \bar{g}) = \det(A + c\mathbf{1})g((A + c\mathbf{1})^{-1}X, X) = I_0 + I_1 c + \cdots + I_{n-1}c^{n-1} \qquad (7)$$

where the operator $A = A(g, \bar{g})$ is given by formula (5), $\mathbf{1}$ denotes the identity operator, $X \in TM^n$, and c is a real parameter. Let us fix a point $x \in M^n$ and define the linear map $x_{\mathcal{I}} : \mathcal{I} \to \mathrm{Symm}(T_x M^n \otimes T_x M^n)$ given by the formula $\mathcal{I} \ni I \to I(x)$.

6.1. Non-degeneracy condition

Assume that the metrics g and $\bar{g}$ are geodesically equivalent. It is evident that at the singular points the metrics g and $\bar{g}$ have multiple common eigenvalues. Moreover, if $x \in \mathcal{S}$, then $\ker x_{\mathcal{I}} \neq 0$.

Following Kiyohara (see [12]) we give the next definition.

Definition 5. *A singular point x is called non-degenerate if for every $I \in \mathcal{I}$ such that $I(x) = 0$ we have $(\partial_\xi I)(x) \neq 0$ for some $\xi \in T_x M^n$.*

Let us fix a singular point x_0 on the manifold and denote by $\phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_n$ the eigenvalues of the operator A. Let $e_1, \ldots, e_n$ be a smooth frame (given in a neighbourhood of the point x_0) such that $Ae_i = \phi_i e_i$ at the point x_0. We have

$$I_c(x_0) = \Phi_c(x_0) \left\{ \frac{e_1^* \otimes e_1^*}{\phi_1 + c} + \cdots + \frac{e_n^* \otimes e_n^*}{\phi_n + c} \right\} \qquad (8)$$

where $\Phi_c(x_0) \stackrel{\text{def}}{=} (\phi_1 + c) \cdots (\phi_n + c)$. Suppose for example that at the point x_0 we have $-\nu = \phi_1 = \cdots = \phi_k < \phi_{k+1} (k \geqslant 2)$. It is easy to see that $I_\nu(x_0) = 0$.

Let us calculate the form $(\partial_\xi I_\nu)(x_0) = (\nabla_\xi I_\nu)(x_0)$, where ∇ denotes an affine connection without torsion and $\xi \in T_{x_0} M^n$. We have,

$$\begin{aligned} \nabla I_c = \nabla(|A + cI|g(A + cI)^{-1}) &= (\nabla|A + cI|)g(A + cI)^{-1} + |A + cI|(\nabla g)(A + cI)^{-1} \\ &+ |A + cI|g\nabla[(A + cI)^{-1}] = |A + cI|\mathrm{trace}[(A + cI)^{-1} \circ \nabla A]g(A + cI)^{-1} \\ &+ |A + cI|(\nabla g)(A + cI)^{-1} - |A + cI|g(A + cI)^{-1}(\nabla A)(A + cI)^{-1} \end{aligned}$$

where we use the formulae $\nabla(\det A) = (\det A)\mathrm{trace}[A^{-1} \circ \nabla A]$ and $\nabla[A^{-1}] = -A^{-1}(\nabla A)A^{-1}$. Therefore,

$$\begin{aligned} (\nabla_\xi I_c)(x_0) = {} & [(\phi_1 + c) \cdots (\phi_n + c)] \left(\sum_{j=1}^{n} \frac{a^j_{j,\xi}}{\phi_j + c} \right) \sum_{j=1}^{n} \frac{e_j^* \otimes e_j^*}{\phi_j + c} \\ & + [(\phi_1 + c) \cdots (\phi_n + c)] \sum_{ij} \frac{g_{ij,\xi}}{\phi_i + c} e_i^* \otimes e_j^* \\ & - [(\phi_1 + c) \cdots (\phi_n + c)] \sum_{ij} \frac{a^j_{i,\xi}}{(\phi_i + c)(\phi_j + c)} e_i^* \otimes e_j^* \end{aligned}$$

$$= (\phi_1 + c)^{k-2}[(\phi_{k+1} + c) \cdots (\phi_n + c)]\{[a^1_{1,\xi} e^*_2 \otimes e^*_2 + a^2_{2,\xi} e^*_1 \otimes e^*_1 - a^1_{2,\xi} e^*_2 \otimes e^*_1 - a^2_{1,\xi} e^*_1 \otimes e^*_2] + \Omega_k + \mathrm{O}(\phi_1 + c)\} \quad (9)$$

where the form Ω_2 vanish. It gives that

$$(\nabla_\xi I_\nu)(x_0) = \begin{cases} 0 & k \geqslant 3 \\ \tilde{A}_{,\xi} & k = 2 \end{cases} \qquad (*)$$

where $\tilde{A}_{,\xi} = [(\phi_3 + \nu) \cdots (\phi_n + \nu)][a^2_{2,\xi} e^*_1 \otimes e^*_1 + a^1_{1,\xi} e^*_2 \otimes e^*_2 - a^2_{1,\xi} e^*_1 \otimes e^*_2 - a^1_{2,\xi} e^*_2 \otimes e^*_1]$.

Lemma 4. *Suppose that the operator $A(g, \bar{g})$ has $m \leqslant n$ distinct eigenvalues $\rho_1, \ldots, \rho_m$ with multiplicities $l_1, \ldots, l_m$, $\sum_i l_i = n$ at the point x_0. It gives an orthogonal splitting $TU(x_0) \cong W_1 \oplus \cdots \oplus W_m$, $\dim W_i = l_i$ of the tangent bundle in a sufficiently small neighbourhood $U(x_0)$ of the point x_0. The operator $A(g, \bar{g})$ preserves the subspaces W_k at each point in U. Moreover, the distributions W_i are integrable.*

Proof of lemma 4. Consider a point x sufficiently near x_0. Denote by $\rho_{k1}(x), \ldots, \rho_{kl_k}(x)$ the eigenvalues which are close to the value ρ_k. Consider the operator $A_k \stackrel{\mathrm{def}}{=} (A(x) - \rho_{k1}(x)) \cdots (A(x) - \rho_{kl_k}(x))$. A_k is a polynomial of A whose coefficients are symmetric functions of the eigenvalues $\rho_{k1}(x), \ldots, \rho_{kl_k}(x)$. Using the integral $\oint_\gamma \lambda^s \frac{\chi_A(\lambda)}{\chi'_A(\lambda)} \mathrm{d}\lambda$ $(s = 1, 2, \ldots)$, where $\chi_A(\lambda)$ denotes the characteristic polynomial of the operator A, it is easy to see that the symmetric functions of the eigenvalues $\rho_{k1}(x), \ldots, \rho_{kl_k}(x)$ are smooth functions of x in a neighbourhood of x_0. Therefore, A_k is a smooth operator in a neighbourhood of the point x_0. We take $W_k \stackrel{\mathrm{def}}{=} \ker A_k$ $(k = 1, \ldots, m)$. Finally, it follows from Levi-Civita's theorem that the distributions W_k are integrable in a neighbourhood of any *stable point*. The set of stable points $\mathcal{M}(g, \bar{g})$ is everywhere dense (see [18]). Thus, W_k are integrable. This completes the proof of lemma 4. □

Proposition 1. *Let the metrics g and $\bar{g}$ be geodesically equivalent. Suppose that x_0 is a singular point and $TU \cong W_1 \oplus \cdots \oplus W_m$ is the corresponding splitting given by lemma 4 in a neighbourhood U of the point x_0; then the singular point x_0 is non-degenerate if and only if one of the next two conditions is satisfied:*

(i) $\dim W_i \leqslant 2$ $(i = 1, \ldots, m)$ *and for every W_p such that* $\dim W_p = 2$ *we have that $(\partial_\xi \tilde{A}_p)(x_0) \neq 0$ for some $\xi \in T_{x_0} M^n$, where $\tilde{A}_p$ denotes the restriction of the operator $A(g, \bar{g})$ to the vector bundle W_p;*

(ii) $\dim W_i \leqslant 2$ $(i = 1, \ldots, m)$ *and for every W_p such that* $\dim W_p = 2$ *the restriction of the function $\Delta_p \stackrel{\mathrm{def}}{=} (\rho_{p1} - \rho_{p2})^2$ to the integral manifold of the distribution W_p which passes through x_0 has a Morse type 'centre' at the point x_0.*

Definition 6. *A non-degenerate singular point x_0 is called non-degenerate singular point of type k if the operator $A(g, \bar{g})$ has exactly k double eigenvalues at the point x_0.*

Proof of proposition 1. Let us take an affine connection ∇ (defined in a neighbourhood of the point x_0) which preserves the distribution W_1. Denote by $\tilde{\nabla}$ its restriction to the vector bundle W_1. It is obvious that $\nabla_\xi A$ preserves the distribution W_1 and $(\nabla_\xi A)_{x_0}|_{W_1} = (\tilde{\nabla}_\xi \tilde{A}_1)_{x_0}$. On the other hand we have that $(\tilde{\nabla}_\xi \tilde{A}_1)_{x_0} = (\partial_\xi \tilde{A}_1)_{x_0}$ and

$$(\nabla_\xi A)_{x_0}|_{W_1} = \begin{bmatrix} a^1_{1,\xi} & a^1_{2,\xi} \\ a^2_{1,\xi} & a^2_{2,\xi} \end{bmatrix}.$$

It follows from $(*)$ that $(\nabla_\xi I_\nu)(x_0) \neq 0$ if and only if $k = 2$ and $(\partial_\xi \tilde{A}_1)(x_0) \neq 0$. Therefore, if x_0 is non-degenerate singular point, then $\dim W_i \leqslant 2$ $(i = 1, \ldots, m)$ and $(\partial_\xi \tilde{A}_p)(x_0) \neq 0$, if $\dim W_p = 2$.

Conversely, suppose that $\dim W_i \leqslant 2$ $(1 \leqslant i \leqslant m)$ and $(\partial_\xi \tilde{A}_p)(x_0) \neq 0$, if $\dim W_p = 2$. Let $\nu_1, \ldots, \nu_k$ be the double eigenvalues of the operator $A(g, \bar{g})$. It can be easily seen that the integrals $I_{-\nu_1}, \ldots, I_{-\nu_k}$ form a basis of the subspace of integrals from $\mathcal{I}$ which vanish at the point x_0. Therefore, if $I(x_0) = 0$, then

$$I = \sum_{i=1}^{k} c_i I_{-\nu_i} \qquad c_i = \text{const}. \tag{10}$$

It follows from $(*)$ and (10) that

$$(\nabla_\xi I)(x_0) = \sum_{i=1}^{k} c_i (\nabla_\xi I_{-\nu_i})(x_0) \neq 0. \tag{11}$$

This completes the proof of (i).

Without loss of generality it can be assumed that $W_1 = \mathrm{span}\langle e_1, \ldots, e_k\rangle$ in a neighbourhood of the point x_0 (lemma 4). In this case we obviously have that $a^i_{j,\xi}(x_0) = (\partial_\xi a^i_j)(x_0)$ $(1 \leqslant i, j \leqslant k)$, where ∂_ξ denotes the partial derivative along the vector $\xi \in T_{x_0}M^n$.

Let us prove (ii). Suppose that x_0 is a non-degenerate singular point and let $\dim W_1 = 2$. It follows from $(*)$ that

$$(\mathrm{sgrad}\, I_\nu)_{x_0} = C_\nu \sum_{i=1}^{n} (\partial_{e_i} a^2_2 P_1^2 + \partial_{e_i} a^1_1 P_2^2 - 2\partial_{e_i} a^1_2 P_1 P_2) \frac{\partial}{\partial P_i} \tag{12}$$

where $P_i \stackrel{\mathrm{def}}{=} \langle P, e_i\rangle$, $P \in T^*M^n$ and $C_\nu \neq 0$. The condition $\{E_g, I_\nu\} = 0$, where $E_g = \sum_i P_i^2$ is the energy integral, gives that $\partial_{e_k} a^i_j = 0$ $(k \geqslant 2)$ and $(\mathrm{sgrad}\, I_\nu)_{x_0} = (\alpha P_1 + \beta P_2)(P_1 \frac{\partial}{\partial P_2} - P_2 \frac{\partial}{\partial P_1})$, α, β are constants. Making orthonormal change of the frame (e_1, e_2) we get that $\alpha = 0$ and $\beta \neq 0$. Therefore, $\partial_{e_1} a^2_2 = \partial_{e_1} a^1_2 = 0$, $\partial_{e_1} a^1_1 = c \neq 0$, and $\partial_{e_2} a^2_2 = \partial_{e_2} a^1_1 = 0$, $\partial_{e_2} a^1_2 = \frac{c}{2} \neq 0$.

We have $\Delta_1 \stackrel{\mathrm{def}}{=} (\rho_{11} - \rho_{12})^2 = (\mathrm{trace}\, \tilde{A}_1)^2 - 4 \det \tilde{A}_1 = (a^1_1 - a^2_2)^2 + 4(a^1_2)^2$. Let us take a curve $\gamma(t)$ $(\gamma(0) = x_0$ and $\dot{\gamma}(0) = \xi_1 e_1 + \xi_2 e_2)$, which is defined in a neighbourhood of the point $t = 0$. We obtain $\frac{\mathrm{d}^2\Delta_1}{\mathrm{d}t^2}(0) = 2((\dot{a}^1_1 - \dot{a}^2_2)^2 + 4(\dot{a}^1_2)^2) = 2c^2(\xi_1^2 + \xi_2^2)$. The last equality proves the item (ii). Proposition 1 is proved. □

Finally combining proposition 1 with corollary 1 and theorem 4 we obtain the following theorem.

Theorem 5. *Suppose that the metrics g and $\bar{g}$ are geodesically equivalent and strictly non-proportional at some point on the manifold; then for every point $x \in M^n$ the multiplicities of the eigenvalues of the operator $A|_x$ are less than or equal to 2. A singular point x_0 is non-degenerate iff the restrictions of the locally defined functions Δ_p on the integral manifolds of the distributions W_p have Morse singularities of index zero (type 'centre') at the point x_0.*

6.2. Local description of the singular points

Here we give a local description of geodesically equivalent metrics in a neighbourhood of a non-degenerate singular point.

Theorem 6. *Suppose that the metrics g and $\bar{g}$ are geodesically equivalent and let x_0 be a non-degenerate singular point of type k; then there exist coordinate system $\{(w_1, \ldots, w_k, y_{2k+1}, \ldots, y_n)\}$, $w_p \stackrel{\mathrm{def}}{=} y_{2p-1} + \mathrm{i}y_{2p}$, $x_0 = 0$, given in a neighbourhood $U(x_0)$ of the point x_0, and a branched covering $\Phi : (D^2)^k \times J^{n-2k} \to U(x_0)$ defined by*

$$\Phi : (z_1, \ldots, z_k, x_{2k+1}, \ldots, x_n) \mapsto (z_1^2, \ldots, z_k^2, x_{2k+1}, \ldots, x_n) \tag{13}$$

where $z_p \stackrel{\text{def}}{=} x_{2p-1} + \mathrm{i}x_{2p}$, *$D^2$ is a two-dimensional disc in $\boldsymbol{R}^2$, and J is an interval. On the covering, we have that*

$$\mathrm{d}g^2 = |\Pi_1|\,\mathrm{d}x_1^2 + |\Pi_2|\,\mathrm{d}x_2^2 + \cdots + |\Pi_n|\,\mathrm{d}x_n^2 \tag{14}$$

$$\mathrm{d}\bar{g}^2 = \rho^1|\Pi_1|\,\mathrm{d}x_1^2 + \rho^2|\Pi_2|\,\mathrm{d}x_2^2 + \cdots + \rho^n|\Pi_n|\,\mathrm{d}x_n^2 \tag{15}$$

$\rho^q \stackrel{\text{def}}{=} \frac{1}{\phi_1\cdots\phi_n}\frac{1}{\phi_q}$, $\Pi_j \stackrel{\text{def}}{=} (\phi_j - \phi_1)\cdots(\phi_j - \phi_{j-1})(\phi_{j+1} - \phi_j)\cdots(\phi_n - \phi_j)$, *where $\phi_j > 0$ are smooth even functions depending only on the variable x_j. These functions satisfy the next conditions*

(a) $\phi_{2j-1}^{(2q)}(0) = (-1)^q\phi_{2j}^{(2q)}(0)$ $j = 1, \ldots, k$;
(b) $\phi''_{2j-1}(0) \neq 0$ $(j = 1, \ldots, k)$ *('non-degeneracy condition').*

Conversely, if g and $\bar{g}$ are defined by formulae (14) and (15), then their projections on $U(x_0)$ are well-defined smooth geodesically equivalent metrics and $x_0 = 0$ is non-degenerate singular point of type k.

Proof of theorem 6. Let us fix a point x_0 on the manifold. Suppose that x is a stable point of type n, i.e., $x \in \mathcal{M}^n$. Let $X_1, \ldots, X_n$ be the principal axes of the operator $A(g, \bar{g})$, which are smoothly defined in a neighbourhood of the point x. Making the Legendre transformation corresponding to the Riemannian metric g we obtain

$$I_c = \Phi_c\left\{\frac{P_1^2}{\phi_1 + c} + \cdots + \frac{P_n^2}{\phi_n + c}\right\} \tag{16}$$

where $\Phi_c \stackrel{\text{def}}{=} (\phi_1 + c)\cdots(\phi_n + c)$ and $P_k = \langle P, X_k\rangle$, $P \in T^*M^n$. Suppose for example that $\phi_1 < \cdots < \phi_n$.

Let $c_1, \ldots, c_n$ be different real numbers such that $\phi_i + c_j \neq 0$. We obviously have

$$\begin{cases} \dfrac{1}{\Phi_1}F_1 = \dfrac{P_1^2}{\phi_1 + c_1} + \cdots + \dfrac{P_n^2}{\phi_n + c_1} \\ \vdots \\ \dfrac{1}{\Phi_n}F_n = \dfrac{P_1^2}{\phi_1 + c_n} + \cdots + \dfrac{P_n^2}{\phi_n + c_n} \end{cases}$$

where $F_k \stackrel{\text{def}}{=} I_{c_k}$.

Lemma 5. *Suppose that* $B \stackrel{\text{def}}{=} (\frac{1}{c_i + \phi_j})_{n\times n}$; *then*

$$\det B = \frac{\prod_{i>j}(\phi_i - \phi_j)\prod_{i>j}(c_i - c_j)}{\prod_{ij}(\phi_i + c_j)}. \tag{17}$$

Denote by B_{lk} the elements of the inverse matrix B^{-1}. We have

$$B_{lk} = (-1)^{l+k}\frac{\prod_{\substack{i>j\\ i,j\neq l}}(\phi_i - \phi_j)\prod_{\substack{i>j\\ i,j\neq k}}(c_i - c_j)}{\prod_{\substack{i\neq l\\ j\neq k}}(\phi_i + c_j)}\,\frac{\prod_{ij}(\phi_i + c_j)}{\prod_{i>j}(\phi_i - \phi_j)\prod_{i>j}(c_i - c_j)} \tag{18}$$

$$= (-1)^{l+k}\frac{1}{\Pi_l}\frac{1}{C_k}\frac{\prod_{ij}(\phi_i + c_j)}{\prod_{\substack{i\neq l\\ j\neq k}}(\phi_i + c_j)} \tag{19}$$

where $\Pi_l \stackrel{\text{def}}{=} (\phi_n - \phi_l)\cdots(\phi_{l+1} - \phi_l)(\phi_l - \phi_{l-1})\cdots(\phi_l - \phi_1)$ and $C_k \stackrel{\text{def}}{=} (c_n - c_k)\cdots(c_{k+1} - c_k)(c_k - c_{k-1})\cdots(c_k - c_1)$. Therefore,

$$P_l^2 = \sum_k (-1)^{l+k} \frac{1}{\Pi_l} \frac{1}{C_k} \frac{\prod_{ij}(\phi_i + c_j)}{\prod_{\substack{i\neq l\\ j\neq k}}(\phi_i + c_j)} \frac{1}{\Phi_k} F_k$$

$$= \sum_k (-1)^{l+k} \frac{1}{\Pi_l} \frac{1}{C_k} \frac{(\phi_l + c_1)\cdots(\phi_l + c_n)}{(\phi_l + c_k)} F_k. \tag{20}$$

Now, suppose that $x_0 \in M^n$ is a non-degenerate singular point of type k $(1 \leqslant k \leqslant [\frac{n}{2}])$. Let $\nu_1 < \cdots < \nu_k$ be the double roots of the characteristic polynomial $\chi_A(\lambda)$ at the point x_0. Suppose in addition that the eigenvalues $\phi_{2s-1} \leqslant \phi_{2s}$ are close to the value ν_s $(s = 1, \ldots, k)$ and $\phi_{2k+1} < \cdots < \phi_n$. The last conditions define ϕ_i $(i = 1, \ldots, n)$ uniquely in a sufficiently small neighbourhood of the point x_0 (see theorem 5).

Let us take $c_{2s-1} = -\nu_s$ $(s = 1, \ldots, k)$ in the formulae above, and suppose as before that $\phi_{2s} + c_{2s} \neq 0$ $(s = 1, \ldots, k)$ and $\phi_l + c_l \neq 0$ $(l = 2k + 1, \ldots, n)$. We obviously have that $F_1(x_0) = F_3(x_0) = \cdots = F_{2k-1}(x_0) = 0$ and the quadratic forms $F_2(x_0), \ldots, F_{2k}(x_0), F_{2k+1}(x_0), \ldots, F_n(x_0)$ are linearly independent.

Suppose that the stable point x is close to the singular point x_0. It follows from (20) that

$$\mathcal{A}_1 = \sum_j (-1)^{1+j} \frac{C_1}{C_j} \left(\frac{\phi_1 + c_1}{\phi_1 + c_j}\right) F_j \tag{21}$$

$$\mathcal{A}_2 = -\sum_j (-1)^{1+j} \frac{C_1}{C_j} \left(\frac{\phi_2 + c_1}{\phi_2 + c_j}\right) F_j \tag{22}$$

$$\vdots$$

$$\mathcal{A}_{2k-1} = \sum_j (-1)^{2k-1+j} \frac{C_{2k-1}}{C_j} \left(\frac{\phi_{2k-1} + c_{2k-1}}{\phi_{2k-1} + c_j}\right) F_j \tag{23}$$

$$\mathcal{A}_{2k} = -\sum_j (-1)^{2k-1+j} \frac{C_{2k-1}}{C_j} \left(\frac{\phi_{2k} + c_{2k-1}}{\phi_{2k} + c_j}\right) F_j \tag{24}$$

$$\vdots$$

$$\mathcal{A}_l = \sum_j (-1)^{l+j} \frac{C_n}{C_j} \left(\frac{\phi_l + c_n}{\phi_l + c_j}\right) F_j \tag{25}$$

where

$$\mathcal{A}_{2s-1} \stackrel{\text{def}}{=} \Pi_{2s-1} C_{2s-1} \frac{(\phi_{2s-1} + c_{2s-1})}{(\phi_{2s-1} + c_1)\cdots(\phi_{2s-1} + c_n)} P_{2s-1}^2 \tag{26}$$

$$\mathcal{A}_{2s} \stackrel{\text{def}}{=} \Pi_{2s} C_{2s-1} \frac{(\phi_{2s} + c_{2s-1})}{(\phi_{2s} + c_1)\cdots(\phi_{2s} + c_n)} P_{2s}^2 \tag{27}$$

$(s = 1, \ldots, k)$ and

$$\mathcal{A}_l \stackrel{\text{def}}{=} \Pi_l C_n \frac{(\phi_l + c_n)}{(\phi_l + c_1)\cdots(\phi_l + c_n)} P_l^2 \; (l = 2k + 1, \ldots, n). \tag{28}$$

The decomposition of the tangent bundle given by lemma 4 gives a Cartesian product structure in a neighbourhood $U(x_0)$ of the point x_0, $U(x_0) \cong D_1^2 \times \cdots \times D_k^2 \times J_{2k+1} \times \cdots \times J_n$, where D_i^2 are 2-discs and J_j are intervals. Suppose that the coordinates q_{2s-1} and q_{2s} refer to the disc D_s^2 $(s = 1, \ldots, k)$, and the coordinate q_l refers to the interval J_l $(l = 2k + 1, \ldots, n)$. Let us set $q_i(x_0) = 0$ $(i = 1, \ldots, n)$.

In an open neighbourhood of the point x_0 the set of singular points $\mathcal{S}$ is a union of submanifolds $\mathcal{S}_{\nu_s} \stackrel{\text{def}}{=} \{\phi_{2s} = \phi_{2s-1}\}$ $(s = 1, \ldots, k)$ of codimension two.

The Stäckel theorem shows that locally, in a neighbourhood of any stable point, the eigenvalues ϕ_{2s-1} and ϕ_{2s} depend only on the variables q_{2s-1} and q_{2s} $(s = 1, \ldots, k)$, and ϕ_l depends only on the variable q_l $(l = 2k+1, \ldots, n)$. It gives that $\mathcal{S}_{\nu_s} = \{q_{2s-1} = q_{2s} = 0\} = \{q | \phi_{2s-1}(q) = \phi_{2s}(q)\}$ $(s = 1, \ldots, k)$.

It follows from (26) and (27) that the forms $\mathcal{A}_{2s-1}$ and $\mathcal{A}_{2s}$ can be extended smoothly to $U(x_0) \setminus \mathcal{S}_{\nu_s}$ $(s = 1, \ldots, k)$. Formula (28) shows that the forms $\mathcal{A}_l$ $(l = 2k+1, \ldots, n)$ can be extended smoothly to all of $U(x_0)$.

It follows from the Stäckel theorem that the forms $\mathcal{A}_i$ $(i = 1, \ldots, n)$ are in involution on $U(x_0) \setminus \mathcal{S}$. Recall that $\mathcal{A}_l = \epsilon_l \langle P, Y_i \rangle^2$, $\epsilon_l = \pm 1$ where the vector fields Y_i are defined uniquely up to multiplication on ± 1. Therefore, the vector fields $\pm Y_i$ are also in involution on $U(x_0) \setminus \mathcal{S}$. Note that $Y_{2s-1}, Y_{2s} \in W_s$ $(1 \leqslant s \leqslant k)$ and $Y_l \in W_l$ $(2k+1 \leqslant l \leqslant n)$. Therefore, the coefficients of the vector fields $\pm Y_{2s-1}$ and $\pm Y_{2s}$ depend only on the variables q_{2s-1} and q_{2s}, and the coefficients of the vector field $\pm Y_l$ depend only on the variable q_l $(1 \leqslant s \leqslant k, 2k+1 \leqslant l \leqslant n)$.

Consider for example the vector fields $\pm Y_1$ and $\pm Y_2$. In every simply connected domain in $D_1^2 \setminus 0$ we can assume that $Y_i = a_i(q_1, q_2)\frac{\partial}{\partial q_1} + b_i(q_1, q_2)\frac{\partial}{\partial q_2}$ $(i = 1, 2)$, where the functions a_i and b_i are defined smoothly, and the frame (Y_1, Y_2) has positive orientation. We suppose that the standard orientation on D_1^2 given by the order of the coordinates (q_1, q_2) is fixed. Let $\imath$ be the linear operator that takes Y_1 to Y_2 and Y_2 to $-Y_1$. It can be easily seen that $\imath$ defines a complex structure on $D_1^2 \setminus 0$. Moreover, it turns out that $\imath$ gives a complex structure on D_1^2. Let us consider the form $G_1 \stackrel{\text{def}}{=} \frac{Y_1^2 + Y_2^2}{\phi_2 - \phi_1}$. The restriction of G_1 to any integral manifold of the distribution W_1 gives a Riemannian metric $\bar{G}_1$ on it (after applying the corresponding Legendre transformation). In coordinates, these metrics depend only on the variables q_1 and q_2. It is evident that the complex structure given by $\bar{G}_1$ coincide with $\imath$. Finally, it can be easily seen that

$$\begin{aligned}\bar{G}_1 &= cg(A - \phi_3)^{-1} \cdots (A - \phi_n)^{-1}(A + c_2) \cdots (A + c_n) \\ &= cg(A^2 - (\phi_3 + \phi_4)A + \phi_3\phi_4)^{-1} \cdots (A + c_n)\end{aligned} \tag{29}$$

where g and A denote the restrictions of the corresponding tensor fields on the integral manifold of the distribution W_1, and c is a constant. Hence, $\bar{G}_1$ is smooth.

Consider the vector fields $Z \stackrel{\text{def}}{=} \pm(Y_1 - \mathrm{i}Y_2)$. This field is well defined and holomorphic (with respect to the complex structure $\imath$ defined above) in a small neighbourhood of every point $x \in D_1^2$, $x \neq 0$. The field Z^2 is defined and holomorphic in the whole D_1^2. Hence, $Z^2 = a^{(q)}(q)\partial_q^2$, $q \stackrel{\text{def}}{=} q_1 + \mathrm{i}q_2$, and the complex-valued function $a^{(q)}(q)$ is holomorphic, $a^{(q)}(0) = 0$. We have that $G_1 = \frac{Z\bar{Z}}{\phi_2 - \phi_1} = \frac{|a^{(q)}(q)|}{\phi_2 - \phi_1}\partial_q \partial_{\bar{q}} > 0$. Therefore, the smooth function $\lambda(q) \stackrel{\text{def}}{=} \frac{|a^{(q)}(q)|^2}{\Delta_1} > 0$ on D_1^2. Suppose that $a^{(q)}(q) = q^k p(q)$, $k \geqslant 2$, and $p(q)$ is a holomorphic function, $p(0) \neq 0$. It follows from theorem 5 that $\Delta_1(q) = \mathrm{O}(|q|^2)$ $(|q| \to 0)$. From another viewpoint we have that $|a^{(q)}(q)|^2 = \mathrm{O}(|q|^{2k})$, $k \geqslant 2$. Therefore, $\lambda(q) \to 0$ $(q \to 0)$. This contradiction shows that $k = 1$, and $a^{(q)}(q) = qp(q)$. Now, it can be easily seen that there exists a biholomorphic change of the variables $w_1 = h(q)$, $w_1 \stackrel{\text{def}}{=} y_1 + \mathrm{i}y_2$, such that the mapping $\Phi_1 : z \mapsto w_1 = z^2$, $z \stackrel{\text{def}}{=} x_1 + \mathrm{i}x_2$, takes ∂_z^2 to Z^2. Hence, $\Phi_{1*}(\partial_{x_i}) = Y_i^2$ $(i = 1, 2)$.

Therefore, there is a coordinate system $\{(w_1, \ldots, w_k, y_{2k+1}, \ldots, y_n)\}$, $w_p \stackrel{\text{def}}{=} y_{2p-1} + \mathrm{i}y_{2p}$, $x_0 = 0$, given in a neighbourhood $U(x_0)$ of the point x_0, and a branched covering Φ :

$(D^2)^k \times J^{n-2k} \to U(x_0)$ defined by

$$\Phi : (z_1, \dots, z_k, x_{2k+1}, \dots, x_n) \mapsto (z_1^2, \dots, z_k^2, x_{2k+1}, \dots, x_n) \tag{30}$$

where $z_p \stackrel{\text{def}}{=} x_{2p-1} + \mathrm{i}x_{2p}$, such that $\Phi_*(\partial^2_{x_j}) = Y_j^2$ $(j = 1, \dots, n)$. Therefore, on the covering, in the coordinates $\{(z_1, \dots, z_k, x_{2k+1}, \dots, x_n)\}$ we have that

$$\begin{aligned} \mathrm{d}g^2 &= P_1^2 + \cdots + P_n^2 \\ &= |\Pi_1 a_1|\,\mathrm{d}x_1^2 + |\Pi_2 a_2|\,\mathrm{d}x_2^2 + \cdots + |\Pi_n a_n|\,\mathrm{d}x_n^2. \end{aligned} \tag{31}$$

It follows from definition of the operator $A(g, \bar{g})$ that

$$\bar{g}(X, X) = \frac{1}{\det A} g(A^{-1}X, X). \tag{32}$$

Hence,

$$\mathrm{d}\bar{g}^2 = \rho^1 |\Pi_1 a_1|\,\mathrm{d}x_1^2 + \rho^2 |\Pi_2 a_2|\,\mathrm{d}x_2^2 + \cdots + \rho^n |\Pi_n a_n|\,\mathrm{d}x_n^2 \tag{33}$$

where $\rho^q \stackrel{\text{def}}{=} \frac{1}{\phi_1 \cdots \phi_n} \frac{1}{\phi_q}$, and the eigenvalues $\phi_j > 0$ and the functions a_j are smooth even functions depending only on the variable x_j. It follows from the smoothness of the conformal multiplier $\lambda(q)$ considered above that these functions satisfy the following conditions:

(a) $\phi_{2j-1}^{(2q)}(0) = (-1)^q \phi_{2j}^{(2q)}(0)$ $(j = 1, \dots, k)$;
(b) $\phi''_{2j-1}(0) \neq 0$ $(j = 1, \dots, k)$ ('non-degeneracy condition').

Remark 3. If we consider the vector fields $Z_1^2 = C_1^{-1}(\phi_1 + c_2) \cdots (\phi_1 + c_n) Y_1^2$ and $Z_2^2 = C_1^{-1}(\phi_2 + c_2) \cdots (\phi_2 + c_n) Y_2^2$ and apply the construction used above, we obtain $a_j \equiv 1$ $(j = 1, \dots, n)$. The conditions on the functions ϕ_j are preserved.

The inverse part of the theorem is obvious. This completes the proof of theorem 6. □

7. Quantum integrability

In the case of geodesically equivalent metrics, classical integrability implies the quantum one. Consider the linear partial-differential operators $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_{n-1}$ given by $\mathcal{I}_k(f) \stackrel{\text{def}}{=} \operatorname{div}(S_k(\operatorname{grad}(f)))$, where $\operatorname{grad}(f)$ denotes the gradient $g^{i\alpha} \frac{\partial f}{\partial x^\alpha}$ of the function f and div denotes the divergence with respect to the metric g.

Remark 4. In coordinates the operators $\mathcal{I}_k$ are given by

$$\mathcal{I}_k = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} (S_k)^i_\alpha \sqrt{\det(g)} g^{\alpha j} \frac{\partial}{\partial x^j}. \tag{34}$$

Remark 5. The operator $\mathcal{I}_{n-1}$ is exactly the Laplacian Δ_g.

Theorem 7 ([21]). *If the metrics g and $\bar{g}$ on M^n are geodesically equivalent then the operators $\mathcal{I}_k$ pairwise commute. In particular, they commute with Δ_g.*

Remark 6. For closed surfaces theorem 7 was essentially proved in [11].

If the metrics are strictly non-proportional then the operators are linearly independent. Then it is possible to separate equation in the Schroedinger equation $\Delta\phi = \lambda\phi$: the equation naturally splits into the system of n one-dimensional Schroedinger equations.

8. Examples: ellipsoid, Euler and Clebsch cases of rigid body motion

Levi-Civita's theorem gives us a series of examples of geodesically equivalent metrics on the torus $T^n = S^1 \times \cdots \times S^1$. We enumerate the circles in the direct product by the numbers $\{1, \ldots, n\}$, let $x_k \in (\boldsymbol{R} \bmod L_k)$, $L_k > 0$, be the coordinate on the kth circle so that $x_1, \ldots, x_n$ is a coordinate system on the torus. Let $\phi_1, \ldots, \phi_n : T^n \to \boldsymbol{R}$ be smooth positive functions on the torus such that for any i the function ϕ_i depends on the coordinate x_i only and $\phi_i < \phi_{i+1}$ (in other words, ϕ_i is a function on the ith circle $(\boldsymbol{R} \bmod L_k)$ and for any $x, y \in \boldsymbol{R}$ the value $\phi_i(x)$ is less than $\phi_{i+1}(y)$). Then the metrics (2), (3) are well definite and are geodesically equivalent on the torus. We will call them *model* metrics on the torus. Each pair of model metrics is given by the positive numbers L_i and by the functions $\phi_i : (\boldsymbol{R} \bmod L_i) \to \boldsymbol{R}$.

The following theorems show that the model metrics give us essentially all possible examples on the torus.

Theorem 8. *Let metrics g, $\bar{g}$ on the torus $\tilde{T}^n$ be geodesically equivalent and strictly non-proportional at least at one point. Then they are strictly non-proportional everywhere.*

This theorem is non-trivial: if the manifold is neither the torus nor is covered by the torus, there must exist points where geodesically equivalent metrics are not strictly non-proportional:

Theorem 9 ([20]). *Let g, $\bar{g}$ be geodesically equivalent metrics on M^n. Suppose that there is no vector field which is Killing for both metrics and that the metrics g, $\bar{g}$ are strictly non-proportional at each point of M^n. Then there exists a covering $\pi : T^n \to M^n$ and a pair g_{model}, $\bar{g}_{\text{model}}$ of model metrics on T^n such that $\pi^* g = g_{\text{model}}$, $\pi^* \bar{g} = \bar{g}_{\text{model}}$.*

We will try to explain theorems 8, 9. Assume that the metrics g, $\bar{g}$ are geodesically equivalent on the torus T^n. Without loss of generality, we can assume that there is no vector field that is Killing with respect to both metrics. More precisely, because of topological reasons, any Killing vector field is zero nowhere.

In Levi-Civita's coordinates (see Levi-Civita's theorem), it is easy to check that a component number i of a Killing (with respect to both metrics) vector field is not zero if and only if it is constant and if the corresponding ϕ_i is constant. In particular, any two Killing (with respect to both metrics) vector fields commute. Therefore the set of all Killing (with respect to both metrics) vector fields generates a free action of the torus T^k on the torus T^n, and the factorspace modulo this action is homeomorphic to the torus T^{n-k} with two geodesically equivalent metrics which admit no Killing (with respect to both metrics) vector field. Thus, we can assume that there is no vector field that is Killing with respect to both metrics, which, in particular, implies, that all singular points are non-degenerate. More precisely, by theorem 4, any eigenvalue of G has multiplicity at most two so that locally the geodesically equivalent metrics behaves as the direct product of at most two-dimensional manifolds with geodesically equivalent metrics. Finally, for two-dimensional manifolds, any singular point of geodesically equivalent metrics (assuming that the metrics admit no Killing vector field) is nondegenerate, see, e.g., [3, p 123], theorem 6.8.

By theorem 4, there exist numbers $\tau_1, \ldots, \tau_{n-1}$ such that for every $x \in T^n$ we have $\phi_i(x) \leqslant \tau_i \leqslant \phi_{i+1}(x)$; moreover, if $\phi_i(x) < \phi_{i+1}(x)$ for any point x then we can choose τ_i in such a way that $\phi_i(x) < \tau_i < \phi_{i+1}(x)$. Consider the elementary symmetric functions $\sigma_i = \sigma_i(-\tau_1, \ldots, -\tau_{n-1})$ and the Liouville fibre

$$L^n \stackrel{\text{def}}{=} \{p \in T^*T^n : I_k(p) = -\sigma_{n-1-k}, k = 0, 1, \ldots, n-1\}.$$

A point of L^n is called *singular* if the differentials $\mathrm{d}I_0, \ldots, \mathrm{d}I_{n-1}$ are linearly dependent at this point. Denote by π the standard projection $T^*T^n \to T^n$. From the explicit formula (4) for the integrals I_k it is easy to extract that the projection of the fibre L^n has no caustics in non-singular points and that the image of singular points coincides with the set of points where for some $i < n$ either $\tau_i = \phi_i(x)$ or $\tau_i = \phi_{i+1}(x)$.

Note that if at each point of the manifold two geodesically equivalent metric are strictly non-proportional then the corresponding fibre L^n is homeomorphic to the torus (by the Arnold–Liouville theorem) and covers the manifold (because it has no caustics), which implies the first part of theorem 9.

By the Arnold–Liouville theorem, the fibre L^n is a union of orbits of the Poisson action of the group $\boldsymbol{R}^n$. Consider any orbit O^n of dimension n of the Poisson action. Let us show that the mapping $\pi_* : H_1(O^n, Q) \to H_1(T^n, Q)$ is a virtual surjection (that is, the image of $H_1(O^n, Q)$ has finite index in $H_1(T^n, Q)$). Take an arbitrary element of $H_1(T^n, Q)$, it can be realized by a closed curve on T^n. We can perturb the curve in such a way that for any point x of the curve, for any $i < n$, either $\phi_i(x) \neq \tau_i$ or $\phi_i(x) = \phi_{i+1}(x)$. Indeed, from theorem 6 it follows that for any $i < n$ the set

$$S_i \stackrel{\text{def}}{=} \{x \in T^*T^n : (\phi_i(x) - \tau_i)(\phi_{i+1}(x) - \tau_i) = 0\}$$

is a submanifold of co-dimension one; from the third part of theorem 4 it follows that any connected component of S_i has a point x where $\phi_i(x) = \tau_i = \phi_{i+1}(x)$. The same argument proves that the image $\pi(O^n)$ contains all points where the metrics are strictly non-proportional.

Denote the perturbed curve by γ. From theorem 6, (and also from Levi-Civita's theorem), it follows that locally the curve γ can be lifted to O^n. Therefore, there exists a curve $\gamma_0 \subset O^2$ such that $\pi(\gamma_0) = \gamma$. We can always assume that the curve γ has at least one point where the metrics are strictly non-proportional. Since the integrals are quadratic in velocities, the number of points of L^n lying over this point is less than or equal to 2^n. Therefore, the curve γ^N, where $N \stackrel{\text{def}}{=} 2n!$, can be realized as the projection of some closed curve from O^n. Thus the the mapping $\pi_* : H_1(O^n, Q) \to H_1(T^n, Q)$ is a virtual surjection.

Finally, the dimension of $H_1(T^n, Q)$ is n, the dimension of $H_1(O^n, Q)$ is no greater than n and is equal to n if and only if O^n is closed; therefore the fibre L^n has no singular points and therefore at each point of T^n the metrics are strictly non-proportional, q.e.d.

Are there examples of geodesically equivalent metrics on simple-connected manifold? The oldest example is due to Beltrami [1]: the standard metric g_{sphere} of the round sphere $S^n \in \boldsymbol{R}^{n+1}$ is geodesically equivalent to the pull-back $\bar{g}_{\text{sphere}} \stackrel{\text{def}}{=} l^* g_{\text{sphere}}$, where

$$l(x) \stackrel{\text{def}}{=} \frac{L(x)}{|L(x)|}$$

where L is a non-degenerate linear transformation of $\boldsymbol{R}^{n+1}$. Till 1998, there was only one other example known on the sphere, see [22].

By theorem 3, in order to check whether a metric with integrable geodesic flow has a geodesically equivalent metric, one should check whether the integrals can be found in the form 4.

It appears that this is the case for the metric of the ellipsoid, for the metric of the Poisson sphere and for the analogues of the Poisson sphere for the Clebsch case of rigid motion. In particular, by theorem 7 we have that the Laplacians of these metrics are integrable in the quantum sense.

Theorem 10 (Topalov and Matveev [29], Tabachnikov [26]). *The restriction of the Euclidean metric*

$$\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_{n+1}^2$$

to the ellipsoid

$$E^n = \left\{(x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}} = 1\right\}$$

is geodesically equivalent to the restriction of the metric

$$\frac{1}{(\frac{x_1}{a_1})^2 + (\frac{x_2}{a_2})^2 + \cdots + (\frac{x_{n+1}}{a_{n+1}})^2}\left(\frac{\mathrm{d}x_1^2}{a_1} + \frac{\mathrm{d}x_2^2}{a_2} + \cdots + \frac{\mathrm{d}x_{n+1}^2}{a_{n+1}}\right)$$

to the same ellipsoid.

In order to show that the integrals for the metric of the Poisson sphere satisfy the conditions in the theorem 3, we use a variant of a well known construction of obtaining families of pairwise commuting functions on homogeneous spaces [4, 7, 15, 27]. Suppose that the Lie group G acts on the manifold M^n. Let us fix a basis $(e_1, \ldots, e_m)$ of the corresponding Lie algebra $\mathfrak{g}$. Consider the functions $P_i \stackrel{\text{def}}{=} -\langle p, X_i(\pi(p))\rangle \in C^\infty(T^*M)$, where $\pi : T^*M \to M$ is the natural projection on M, $p \in T^*M$ and X_i $(i = 1, \ldots, m)$ are the fundamental vector fields $X_i(x) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}[\exp(te_i) \circ x]$. It can easily be checked (in the case of left action) that $[X_i, X_j] = -c_{ij}^k X_k$, where c_{ij}^k denote the structural constants of the Lie algebra $\mathfrak{g}$. Therefore, $\{P_i, P_j\} = c_{ij}^k P_k$, where $\{,\}$ denotes the canonical Poisson structure on T^*M. Denote by μ the moment map $\mu(p) \stackrel{\text{def}}{=} (P_1(p), \ldots, P_m(p))$, $\mu : T^*M \to \mathfrak{g}^*$.

It is known that if the functions $F, H \in C^\infty(\mathfrak{g}^*)$ are in involution with respect to the standard Lie–Poisson bracket on $\mathfrak{g}^*$, then the functions μ^*F and μ^*H are in involution with respect to the canonical Poisson structure on T^*M. If $F \in C^\infty(\mathfrak{g}^*)$ is polynomial of degree k, then μ^*F is polynomial in momenta of the same degree.

Corollary 4. *Consider a family of quadratic forms* $Q_\lambda(m) = \sum_{k=0}^{n-1}\lambda^k Q_k(m)$ $(m \in \mathfrak{g}^*)$, *which are in involution with respect to the Lie–Poisson bracket on* $\mathfrak{g}^*$. *Suppose that the forms* $\mu^*Q_0, \ldots, \mu^*Q_{n-1}$ *are linearly independent at some point* $x_0 \in M$ *and let the form* μ^*Q_{n-1} *be definite. Suppose in addition that there is an open dense subset* $U \subset M$ *such that*

$$(\mu^*Q_\lambda)(p) = r(\lambda, \pi(p))R(\lambda, p) + c(\lambda)(S(\lambda, p))^2 \qquad p \in T^*U$$

where $r = r(\lambda, \pi(p)) \in C^\infty(\boldsymbol{R} \times U)$ *is an unitary polynomial in* λ *of degree* n *that has* n *different real roots* $\lambda_i = \lambda_i(x)$ $(i = 1, \ldots, n)$ *on* U, $c \in C^\infty(\boldsymbol{R})$, $c(\lambda_i(x)) \neq 0$ $(x \in U)$, *and the functions* R *and* S *are supposed to be smooth on* $(\boldsymbol{R} \setminus N) \times T^*U$, *where* N *is a finite set and* $\lambda_i \in \boldsymbol{R} \setminus N$ $(i = 1, \ldots, n)$; *then*

(a) the metric given by the quadratic form μ^*Q_{n-1} *is geodesically equivalent to the metric given by the quadratic form*

$$\bar{g}_{n-1}(\lambda) = \left(\frac{1}{r(\lambda)}\right)^2 \mu^*Q_\lambda$$

(b) the metric given by the quadratic form μ^*Q_0 *is geodesically equivalent to the metric given by the quadratic form*

$$\bar{g}_0(\lambda) = \left(\frac{r(0)}{r(\lambda)}\right)^2 \mu^*Q_\lambda.$$

This corollary is proved in [32]. In some details we follow this paper.

Let us apply corollary 4 to the multi-dimensional analogue of the Euler case of rigid body motion. For this case, the quadratic form is given by $H_b \stackrel{\text{def}}{=} \sum_{0\leqslant i<j\leqslant n}\frac{X_{ij}^2}{b_ib_j} \in C^\infty(so(n+1)^*)$,

where $X_{ij} \stackrel{\text{def}}{=} E_{ij} - E_{ji}$ $(i < j)$ form a basis of the Lie algebra $so(n+1)$, $b_i \neq b_j$ $(i \neq j)$, $b_i \neq 0$.

Consider the family of metrics $T_{b,c} \stackrel{\text{def}}{=} \sum_{i<j} \frac{c_i - c_j}{b_i - b_j} X_{ij}^{*\,2}$, $c_i \neq c_j$ $(i \neq j)$, that corresponds to the *normal series of sectional operators* (see [7]). We identify the Lie algebra $so(n+1)$ with the dual space using the Killing form. The Hamiltonian systems $\dot{x} = \mathrm{ad}_{\nabla_x T_{b,c}} x$ are completely integrable ($\nabla_x f$ denotes the gradient of the function f calculated with respect to the Killing form on $so(n+1)$). If b_i $(i = 0, \ldots, n)$ are fixed, then the corresponding integrals can be taken independent of c_i $(i = 0, \ldots, n)$ (we can take, for example, the integrals obtained from the shift argument method (see [7])). Therefore, the forms $T_{b,c}$ (b is fixed) are in involution with respect to the Lie–Poisson bracket. Finally, taking $c_i = \frac{1}{b_i - \lambda}$ (λ is a real parameter), we obtain a family of pairwise commuting functions on $so(n+1)^*$

$$Q_\lambda \stackrel{\text{def}}{=} \prod_{k=0}^{n} (b_k - \lambda) \left\{ \sum_{i<j} \frac{X_{ij}^2}{(b_i - \lambda)(b_j - \lambda)} \right\}.$$

We have

$$Q_0 = (\Pi_{k=0}^{n} b_k) \sum_{i<j} \frac{X_{ij}^2}{b_i b_j}$$

and

$$Q_{n-1} = (-1)^{n+1} (\Pi_{k=0}^{n} b_k) \sum_{i<j} X_{ij}^2.$$

The form Q_0 coincide (up to multiplication on a constant) with H_b and Q_{n-1} coincide with the Killing form on $so(n+1)$.

Consider the standard left action of the group $SO(n+1)$ on $\boldsymbol{R}^{n+1}$ supplied with the Euclidean metric $\mathrm{d}g_E^2 \stackrel{\text{def}}{=} \mathrm{d}x_0^2 + \cdots + \mathrm{d}x_n^2$. This action gives an action on the unit sphere S^n.

By [12] we have that

$$\mu^* Q_\lambda = \prod_{k=0}^{n} (b_k - \lambda) \left\{ \left(\sum_{i=0}^{n} \frac{x_i^2}{b_i - \lambda} \right) \left(\sum_{i=0}^{n} \frac{\partial_i^2}{b_i - \lambda} \right) - \left(\sum_{i=0}^{n} \frac{x_i \partial_i}{b_i - \lambda} \right)^2 \right\}$$

where ∂_i denotes the partial derivative $\frac{\partial}{\partial x_i}$. By corollary 4, we obtain the folowing theorem.

Theorem 11 ([32]). *The metric of the Poisson sphere* $\mathrm{d}g_{\text{Poisson}}^2 = \mu^* Q_0$ *is geodesically equivalent to the metric given by the quadratic form*

$$\bar{g}_0(\lambda) = \left(\sum_{i=0}^{n} \frac{x_i^2}{b_i} \right)^2 \bar{g}_{n-1}(\lambda)$$

where

$$\bar{g}_{n-1}(\lambda) = \left(\sum_{i=0}^{n} \frac{x_i^2}{b_i - \lambda} \right)^{-1} \left(\sum_{i=0}^{n} \frac{p_i^2}{b_i - \lambda} \right) - \left(\sum_{i=0}^{n} \frac{x_i^2}{b_i - \lambda} \right)^{-2} \left(\sum_{i=0}^{n} \frac{x_i p_i}{b_i - \lambda} \right)^2$$

and λ is a real parameter.

Remark 7. We identify the cotangent bundle to the unit sphere $T^* S^n$ with the subset

$$\left\{ \sum_{i=0}^{n} x_i^2 = 1, \sum_{i=0}^{n} x_i p_i = 0 \right\} \hookrightarrow T^* \boldsymbol{R}^{n+1}. \tag{35}$$

The embedding is symplectic.

Let us apply corollary 4 to the multi-dimensional analogue of the Clebsch case obtained by Perelomov in [24, 25]. The corresponding systems are Hamiltonian with respect to the standard Lie–Poisson bracket on the dual space of the Lie algebra $e(n)$. The Hamiltonians are given by the formula

$$H_{b,c} \stackrel{\text{def}}{=} \sum_{1\leqslant i<j\leqslant n} \frac{c_i - c_j}{b_i - b_j} X_{ij}^2 + \sum_{i=1}^{n} \frac{c_i - c_{n+1}}{b_{n+1}} Y_i^2 \tag{36}$$

where $b_i \neq b_j$ $(i \neq j)$, $b_{n+1} \neq 0$ and $X_{ij} \stackrel{\text{def}}{=} E_{ij} - E_{ji}$ $(i < j)$, $Y_i \stackrel{\text{def}}{=} E_{i\,n+1}$ form a basis of $e(n)$. If b is fixed, then the Hamiltonians $H_{b,c}$ are completely integrable and the family of integrals can be taken independent of c_i (see [7]). Therefore, the functions $H_{b,c}$ (b is fixed) are in involution with respect to the Lie–Poisson bracket on $e(n)^*$. Taking $c_i = \frac{1}{b_i-\lambda}$ $(i = 1, \dots, n)$, $c_{n+1} = 0$ and $b_{n+1} = 1$, we obtain a family of pairwise commuting quadratic forms

$$Q_\lambda \stackrel{\text{def}}{=} \prod_{k=1}^{n}(b_k - \lambda)\left\{\sum_{i<j} \frac{X_{ij}^2}{(b_i-\lambda)(b_j-\lambda)} - \sum_{i=1}^{n} \frac{Y_i^2}{b_i-\lambda}\right\}. \tag{37}$$

Consider the standard left action of the group of Euclidean transformations $E(n)$ on $\boldsymbol{R}^n$ supplied with the Euclidean metric g_E. Let us take the one-parameter family of pairwise commuting functions on $e(n)^*$ given by formula (37). We have

$$\mu^* Q_\lambda = \prod_{k=1}^{n}(b_k - \lambda)\left\{\left(\sum_{i=1}^{n} \frac{x_i^2}{b_i-\lambda} - 1\right)\left(\sum_{i=1}^{n} \frac{\partial_i^2}{b_i-\lambda}\right) - \left(\sum_{i=1}^{n} \frac{x_i \partial_i}{b_i-\lambda}\right)^2\right\}.$$

Applying corollary 4 we obtain the following theorem.

Theorem 12 ([32]). *The metric of the analogue of the Poisson sphere corresponding to the Clebsch case of motion of the rigid body* $\mathrm{d}g^2_{\text{Clebsch}} = \mu^* Q_0$ *is geodesically equivalent to the metric given by the quadratic form*

$$\bar{g}_0(\lambda) = \left(\sum_{i=1}^{n} \frac{x_i^2}{b_i} - 1\right)^2 \bar{g}_{n-1}(\lambda)$$

where

$$\bar{g}_{n-1}(\lambda) = \left(\sum_{i=1}^{n} \frac{x_i^2}{b_i-\lambda} - 1\right)^{-1}\left(\sum_{i=1}^{n} \frac{p_i^2}{b_i-\lambda}\right) - \left(\sum_{i=1}^{n} \frac{x_i^2}{b_i-\lambda} - 1\right)^{-2}\left(\sum_{i=1}^{n} \frac{x_i p_i}{b_i-\lambda}\right)^2$$

and λ *is a real parameter.*

Finally, combining the results of this section with theorem 7 we obtain the following theorem.

Theorem 13. *The Laplace–Beltrami operator corresponding to the metrics of the ellipsoid, the Poisson sphere and the analogue of the Poisson sphere corresponding to the Clebsch case of motion of the rigid body are completely integrable in the quantum sense.*

Acknowledgments

The authors use the opportunity to thank all participants of the workshop 'Mathematical Methods of Regular Dynamics' for the interest to the problem and for fruitful discussions. The first author is grateful to European Post-Doctoral Institute for financial support and to Warwick University and the Isaac Newton Institute for hospitality. The second author was partially supported by MESC grant MM-810/98. This paper was written during his stay at the University of Nantes supported by CNRS.

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